The Physics of High Temperature Superconductors

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Abstract

We review the physics of high temperature superconductors. We start by looking at the experimental facts and giving a general overview of the cuprate superconductors. Using this as motivation, we follow Anderson’s ideas for constructing a coherent theoretical framework. Our discussion is guided by a set of Central Dogmas based on experimental observation that we use to constrain the structure of any theory of the cuprates. We make speculations about a possible driving force for the large critical temperatures, and present possible pairing mechanisms.
1 Introduction

The purpose of this paper is to give an outline of the physics of high temperature superconductivity. Discovered in 1986, high temperature superconductors are a new class of material, vastly different from conventional superconductors, with critical temperatures as high as 120 K, a factor of 10 higher than is typically seen in conventional materials. In contrast to conventional superconductivity, which evolves out of a simple Fermi-liquid metallic state, the parent of the superconducting compound is an insulating metal oxide, which becomes superconducting at low (5 to 10 %) doping levels. The crystal structure of these materials all involve planes of CuO$_2$ separated by donor metal oxide layers. The mobile electrons in the system are in the copper orbitals in the CuO$_2$ planes.

One of the most important features of these materials in that there are repulsive interactions between the electrons that are very strong, with energy scales on the order of 1000 K. In conventional superconductors, the electron-electron interaction above $T_c$ plays a minor role: the physics of the normal state in conventional superconductors is well described by Fermi-liquid theory, where interactions lead only to a renormalization of dynamical constants, and where low lying excitations still resemble those of the free electron gas. The important idea is that the excitations are well defined quasiparticles, which evolve adiabatically from the the excitations of the non interacting Fermi gas as interactions are turned on.

In the high $T_c$ materials, however, the enormous interaction energy will mean that interactions will play an important role in both the supercon-
ducting state and the normal state. A good example of this is the undoped parent compound: looking at the single particle structure in the planes, the system has a half filled band, and we would predict that the material should be a metal. Experimentally, it is an insulator. The reason for this is electron-electron interactions: roughly, the large coulomb repulsion between two electrons on the same copper orbital strongly disfavors hopping of an electron to an already occupied site. This opens up a gap in the electronic structure, which we see only when we include many body effects. The material is called a Mott insulator, after Mott who first derived this result for a lattice of hydrogen atoms[1].

In addition to the strength of the interaction, the reduced dimensionality of the CuO$_2$ planes will also play an important role. Although we suggested above that the large size of the interaction will modify the physics of our normal state, Fermi liquid theory is not limited to weak interactions, and generally Fermi liquid theory works quite well for repulsive interactions that are large in 3D systems. The argument is that due to the large scattering phase space, even strong interactions in 3D lead to a very dilute gas of quasiparticle excitations. This idea is expressed J. Voit’s papers on luttinger liquids[2]: the correlations in the electron system in 3D are weak, although the interactions may be strong. Dimensionality plays an important role: in 1D, the Fermi liquid becomes unstable for arbitrarily small interactions: correlations in 1D are strong even for weak interactions.

The problem in 1D is not completely lost though: there exists a universal behavior of the solutions of the 1D interacting problem, analogous to the canonical 3D Fermi liquid, called the Tomonaga-Luttinger liquid (gen-
erally referred to as just the Luttinger liquid). The Luttinger liquid is an
exact solution of the Luttinger model, analogous to the exactly solvable free
electron gas from which Fermi liquid theory starts. Luttinger-liquid the-
ory can be thought of as stating that the low energy excitations of any 1D
interacting have a direct correspondence to the low energy excitations of
the Luttinger liquid. The most important feature of the Luttinger liquid is
that the elementary excitations are no longer electron-like: instead, spin and
charge are carried by separate collective excitations with linear dispersion
relations, referred to as spinons and holons. They propagate with different
velocities, leading to what is referred to as “spin-charge separation”. This
can be seen very elegantly in simulations[3]: injecting an electron into a 1D
Luttinger liquid, we see separate peaks for the charge and spin densities that
propagate with different velocities and spread out over time.

Two dimensions appears to be a critical dimension, where we cross over
from one type of universal behavior to another. The nature of the full
solutions of the interacting 2D electron gas are still unknown: we have to
analytic solutions in 2D, as we have in 3D and 1D. Questions concerning
whether the 2D problem shows spin-charge separation, and whether the
Fermi-liquid quasiparticle residues $Z$ are zero, are still open and controversial
questions. Given the exotic behavior of the cuprates though, our suspicion
is that the strong electron interactions and the reduced dimensionality of
the CuO planes will lead to a strongly correlated soup of electrons, which
may fall outside the realm of Fermi liquid theory.

One of the problems with the high $T_c$ problem is that it is really hard.
Experimentally, the materials are very difficult to work with. They are
brittle transition metal oxides that are highly anisotropic. Making reliable
electrical contacts to them can be difficult. It has now become clear that
high purity single crystals are essential in order to obtain reliable, repeatable
measurements. Their surfaces are also complicated: when they cleave, they
do so between interstitial layers, and so tunneling experiments are never
directly into the CuO plane. Theoretically, it is found that there are no
small parameters in the problem: perturbation theory is frequently of little
use, and one is left to search for tricks that will make the problem tractable
without changing the physics. It is for these reasons that the problem has
failed to converge onto a solution despite 15 years of enormous experimental
and theoretical effort.

Since this is a review paper on the physics of high temperature super-
conductors, we will inevitably have to discuss the problem in the context of
some theory for what is going on. Theories of high $T_c$ are a dime a dozen:
there are quite possibly as many theories as there are theorists working on
the problem. The theory I will choose to present is that of P.W. Anderson[4],
as expressed in his modestly titled book “THE Theory of Superconductivity
in the High-$T_c$ Cuprates”.

Anderson’s approach to the problem carries a lot of merit. At the core of
his reasoning is the concept of what he calls the “Central Dogmas” of high
$T_c$. There are logical deductions from experimental facts that constrain the
overall structure of any description of the actual mechanism. It is important
to note that the purpose of the Dogmas is not to entirely determine the
theory, but instead to limit the discussion to theories and mechanisms that
are consistent with logic and the overall burden of experimental fact. High
$T_c$ is a rich and complex field: in solving any complex problem, we must make simplification that will make the problem tractable. In doing this, we must also be careful not to acquire tunnel vision: we must always keep in mind the big picture, and any theory must be consistent with all of the basic overall facts about the cuprates.

2 Overview of the Properties of the Cuprates

2.1 Crystal Structure of the High $T_c$ Cuprates

The basic structure of the cuprate superconductors is a CuO$_2$ plane separated by intervening planes composed of metal donor ions and oxygens. The simplest of these is the La$_{2-x}$Sr$_x$CuO$_4$, and related materials. This structure is shown in Fig. 1: it consists of single atomic planes of CuO$_2$ separated by two atomic planes of La-Sr oxide. This material has a maximum $T_c$ of 38 K at an optimal doping of $x = 0.15$. This was the the first high $T_c$ material discovered.

Not long after, it was discovered that materials that have two or more tightly packed planes, separated by by only one donor-oxide layer, showed considerably higher critical temperatures: these include YBa$_2$Cu$_3$O$_7$ with $T_c$ near 95 K, and Bi$_2$Sr$_2$CaCu$_2$O$_8$ with $T_c$ around 80 K. Both of these have paired planes. Introducing more coupled planes tends to lead to non uniform doping and $T_c$ doesn’t increase significantly. It should be noted that the YBCO structure also has a “chain” layer which was once thought to be important. The general consensus now, though, is that these play only a minor role.
Figure 1: The crystal structure of La$_2$CuO$_4$. (a) The arrows on the coppers denote the 3D orientation of the spins in the antiferromagnetic state. At low temperatures, the materials undergo a transition to an orthorhombic structure, involving a rotation of the Cu-O octahedron as shown, buckling the planes. (b) shows the copper and oxygen orbitals in the plane. Taken from [5].
There are also many more complicated structures that give slightly higher \( T_c \)s. The current record is the compound \( \text{HgBa}_2\text{Ca}_2\text{Cu}_3\text{O}_{8+\delta} \) with a \( T_c \) of 135 K. These materials, however, are difficult to manufacture, and high quality single crystals are generally not available. The current focus in the research community has been towards producing high quality materials and performing accurate, reproducible experiments.

### 2.2 Orbital structure in the plane

As we have mentioned before, the important electronic structures in the cuprates are the copper and oxygen orbitals in the planes. To understand the band structure of the \( \text{CuO} \) planes, we will work in the context of a LCAO (linear combination of atomic-like orbitals) picture. We first consider the energy level shifts due to the coupling of a single \( \text{Cu} \) to it’s oxygen neighbors, giving us the band center energies. Adding periodicity, we allow the bands to delocalize: the important detail is that the splitting of the band centers due to hybridization in the \( \text{CuO} \) units will be large compared to the delocalized bandwidth for the important orbitals, and we will end up with an effective model with one band.

The filling of the valances of the atoms in their atomic state and in the ionic states they end up in is as follows:

- \( \text{Cu}: \ 4s^23d^9 \quad \text{Cu}^{2+}: \ 3d^9 \)
- \( \text{O}: \ 2s^22p^4 \quad \text{O}^{2-}: \ 2s^22p^6 \)
- \( \text{La}: \ 6s^25d^1 \quad \text{La}^{3+}: \ (\text{Xe}) \)

The two \( \text{La} \) each give up 3 electrons to the oxygens, and the last oxygen
gains its two electrons from the Cu. Our starting point is one where oxygens all have full shells, and the Cu have 9 electrons in their d shell with an empty s shell. The Cu 3d and O 2p levels start of about an eV or less apart. The coupling matrix elements $V_{dp}$ are large, on the order of a couple of eVs.

In the plane, the coupling is strongest between the oxygen 2p$_\sigma$ orbitals directed along the bonds on the square lattice, and the 3d$_{z^2}$ and 3d$_{x^2-y^2}$ orbitals. Note that we start out with 9e$^-$ on the Cu with 5 levels, and 6e$^-$ on the O with 3 levels: ie. at the end, the highest energy level will have one hole, with everything else filled. The coupling matrix elements of the 3d$_{xy,yz,zx}$ to the 2p$_\sigma$ levels is smaller by about a factor of two, and the 2p$_\pi$ levels on the oxygens only couple weakly. Thus any hybridization involving the O 2p$_\pi$, 2p$_z$ or the Cu 3d$_{xy,yz,zx}$ orbitals can be ignored since the level splitting of these orbitals will be small, and consequently all of the resulting hybrids will be filled bands. Note we have left the Cu 4s orbitals out of the picture: this is justified, because although they are not far off in energy from the 3d, they are at a much larger radius, and so they hybridize strongly and are pushed out of the diagram.

This leaves us with the Cu 3d$_{z^2}$, 3d$_{x^2-y^2}$, and the O 2p$_\sigma$ levels. We will expect that the coupling to the 3d$_{x^2-y^2}$ will be stronger, as it’s “lobes” extend further out into the 2p orbitals than does the “donut” of the 3d$_{z^2}$ orbitals. This is the case in the cuprates, and is evidenced by the fact that the cuprates tend to undergo a “Jahn-Teller” distortion, a deformation of the crystal lattice that such that the CuO planes distort into a square planar structure. This is driven by the filled 3d$_{x^2-y^2}$ bonding orbitals, which gain energy by increasing coupling to the 2p, and pushes the 3d$_{x^2-y^2}$ antibonding
orbitals up further in energy. In the end, the $3d_{x^2-y^2}$ antibonding orbitals are above the $3d_{z^2}$ antibonding ones by a couple of eV.

The resulting picture of the hybrid levels is shown in Fig. 2. The level splitting is larger than the bandwidth we get from delocalizing the states, and we are left with a single band theory.

As we mentioned in the introduction, at this stage we would predict the undoped compound should be a metal, with a half filled band. However, there is a large Coulomb energy that we have to pay to doubly occupy a site, due to the localized nature of the orbitals. This opens up a “charge-gap” in the system at half filling in the $3d_{x^2-y^2}$ antibonding band. We can understand this in a naive picture in the following way: when we delocalize the $3dX_{x^2-y^2}$ antibonding orbital, we form spectrum of linear combinations of the orbitals with varying phases between sites: the top half of the band have an anti-bonding-like phase modulation, and involves some net double occupation. When we add in a mean-field penalty for double occupation
from the many body physics, a gap opens in the spectrum. A quantitative account of how this happens and of general metal-insulator phenomena is covered in detail in reference [6].

2.3 The Effective Hamiltonian

To build any kind of theory, we need to start with an effective Hamiltonian for the system. Because of the large repulsive interactions, a good starting point is the single band Hubbard Model,

\[ \mathcal{H} = -t \sum_{\langle i,j \rangle, \sigma} c_{i \sigma}^\dagger c_{j \sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \]  

where \( c_{i \sigma}^\dagger \) is the creation operator for an electron at site \( i \) with spin \( \sigma \), and \( n_{i \sigma} \) is the number operator for site \( i \) of the square lattice. In this equation, the parameter \( t \) represents the hopping matrix element for an electron to jump to a nearest neighbor site. In second term, the parameter \( U \) represents the large repulsion energy associated with doubly occupying a site. Typically, the cuprates are thought to be in an intermediately coupled regime, with \( U \sim 8t \). The one-band Hubbard model is generally regarded as the simplest model of the CuO planes that captures the important physics.

At low temperatures, we will expect that double occupancy will be strongly disfavored. In this situation, the Hubbard model can be mapped to another one-band effective Hamiltonian, referred to as the \( t-J \) model:

\[ \mathcal{H} = -P \sum_{\langle i,j \rangle, \sigma} t c_{i \sigma}^\dagger c_{j \sigma} P + J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \]  

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Here, $\vec{S}_i$ is the spin of the electron on site $i$, and the operator $P$ projects a state onto zero or single occupancy. The energy $J$ is related to the Hubbard model parameters by $J = \frac{2t^2}{U}$, and comes from the lowering of the kinetic energy that we can obtain by forming singlet pairs of electrons in two neighboring orbitals. Realistic values for the parameters of the Hubbard and $t-J$ models are [8]:

$$U \simeq 5 \text{ eV}$$
$$t \simeq 0.4 \text{ eV}$$
$$J \simeq 0.13 \text{ eV}$$

### 2.4 Phase Diagram of the Cuprates

A generalized phase diagram of the cuprates is shown in Fig. 3. The undoped parent compound of the cuprates is a Mott insulator, as discussed in Section 2.2. There is a strong antiferromagnetic coupling of the spins on intervening copper sites in the plane through the superexchange mechanism:

$$\mathcal{H}_J = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$  \hspace{1cm} (3)

with $J = 135 \text{ meV}$, or around 1000 K. In 3D, we would expect an AF transition at temperatures comparable to $J$. However, the larger fluctuations present in 2D are predicted to drive the ordering temperature of the 2D square Heisenberg model to $T = 0$. In the cuprates, the weak 3D coupling leads to a finite transition temperature at 300K. Note that there is no ordering observed in the plane until the point of the full 3D transition.
The cuprate materials are generally made superconducting by doping holes into the plane by substituting elements in the interstitial layers: for example, in La$_2$CuO$_4$, one replaces La$^{3+}$ with Sr$^{2+}$, leaving one less electron in the CuO plane. As we do this, the AF order is quickly destroyed: at $x \approx 0.02$, the cuprates show no long range AF order. The reason for this rapid destruction of the AF state is that when an electron is remove from the plane, the exchange coupling is changed from strongly antiferromagnetic to ferromagnetic, which quickly frustrates the spin lattice. The leads to a spin glass state between $x = 0.02$ and $x = 0.05$, where short range order in the AF order parameter is observed.

As we move upwards in temperature in the $0.02 \leq x \leq 0.05$ region, we find the material is a poor metal showing some anisotropy. Here we
encounter another mysterious feature referred to as the “spin gap”. In this region, it is seen that low lying spin excitations of the system are suppressed. It is important to mention that the density of states does not go entirely to zero, but is significantly reduced. It is not a true gap as seen in a BCS superconductor or in a semiconductor, and is usually refereed to as a “pseudogap”. There is still much controversy over interpretation of the spin gap, and also about the possibility of an accompanying charge pseudogap.

At doping levels between $x = 0.05$ and $x = 0.30$, we find the superconducting “hump”, with the optimal $T_c$ found around $x = 0.15$. In this region, the superconducting gap is observed to be around 30 or 40 meV, from tunneling and infrared data. The gap shape is roughly similar to the BCS gap, but shows different peak shapes, and characteristic kinks. Normal-Superconductor tunneling also show a reproducible background slope, favoring the tunneling of holes. Also, there does not seem to be a systematic correspondence between the gap and the critical temperature. This strongly contradicts conventional BCS theory: in BCS superconductors, the gap and critical temperature are related through:

$$\frac{\Delta(0)}{kT_c} = 1.764 \quad \text{(BCS)} \quad (4)$$

This is seen to be in agreement with a wide range of conventional superconductors, where $T_c$ varies from less than 1K up to 10 K. In BCS, all of the condensate energy is derived from the pairing interaction: the pairing energy is the only relevant energy, and sets the energy scale for onset superconductivity. This does not seem to be the case in high temperature
superconductors. Fig. 4(a) shows Superconductor-Insulator-Superconductor junction measurements of the gap in BISCCO samples with varying doping. As seen in Fig. 4(b), the tunneling gap does not scale as predicted by Eq. 2.4. Within a compound, generally the gap is seen to increase almost linearly with decreasing doping, even through the hump where the critical temperature is decreasing. The gap is also always larger than the value predicted from Eqn. 2.4.

Above $T_c$, referred to as the “normal” state, a very peculiar metal is found, with large anisotropy, a very characteristic temperature dependence, and seemingly little contribution to the resistivity from impurity and phonon scattering. The out-of-plane resistivity is seen to be a factor of 100 or more...
larger than the in-plane resistivity: for this reason, it is generally referred to as a 2D metal. As this forms the electron liquid that the superconducting state evolves from, it’s properties will be important in understanding superconductivity.

Finally, as we overdope the compound, superconductivity disappears. Anisotropy in the normal state is no longer seen, and the system becomes a 3D metal.

2.5 Resistivity Measurements

One of the most striking features about the cuprate superconductors is the behavior of the resistivity of the normal state that is found above the transition temperature of the optimally doped materials.

Before we look at the resistivity of the high $T_c$ compounds, it will be instructive to first review the properties of the resistance of normal metals. In normal metals, the resistance takes on the form

$$\rho = \rho_{res} + AT^2 + \rho_{ph}$$  \hspace{1cm} (5)

$\rho_{res}$ is a temperature independent resistivity due to impurity scattering. The $T^2$ term arises due to electron-electron scattering. At a temperatures $T$, the density of final states will include scattering off of the empty quasiparticle states found within $kT$ of the Fermi surface. For scattering into a shell of states within an energy $\xi$ of the Fermi surface, the scattering rate will be proportional to the density of final states (Fermi’s golden rule), and the density of states will be proportional to $\xi^2$ for $\xi$ small (see [9] page 346,
Figure 5: The phonon contribution to the resistivity in normal metals. $\theta$ is the Debye temperature. Taken from [11].

and [10] page 92). This leads to the $T^2$ dependence, which we expect to see in systems with strong electron-electron scattering. The last term is the phonon contribution.

The temperature dependence of the phonon contribution is shown in Fig. 5. For $T \ll \theta_D$, there is a contribution $\rho_{ph} \sim T^5$. The strong suppression at low temperatures is due to the low thermal occupation of the phonon modes and the fact that at low temperatures, phonon scattering is strongly peaked in the forward direction due to the presence of the Fermi surface. For temperatures $T > \theta_D$, the contribution is $\rho_{ph} \sim T$, since the thermal occupation of the phonon modes is linear in $T$ at high temperatures. Gen-
erally, it is found that non linear behavior begins around $T \sim 0.3 \theta_D$, and becomes sublinear.

Another important aspect of transport in conventional metals is that the conductivity is generally very isotropic: anisotropies in the material lead to anisotropic effective masses, but this does not lead to anisotropic conductivity. The reason for this is that the velocity corrections will tend to cancel with the density of states corrections, and the conductivity will be much more isotropic than the effective mass. The general result is that the numerical values of the conductivity in a Fermi liquid are not renormalized, and are independent of complicated band and interaction effects. As Anderson points out in his book (see [4] pg. 61), an excellent example of this is the Mott conductivity as the threshold for the metal-insulator transition. This is predicted to have the value

$$\frac{e^2}{h} = \frac{1}{25000} \Omega$$

independent the details of the material, a fact that is confirmed by experiment. The Mott conductivity is determined simply as a criterion as to whether electrons can coherently propagate in plane wave (or alternatively, Bloch) states. There do exist materials that show large anisotropies in conductivity, and in these systems the electrons are found to be in a low dimensional, highly correlated state such as a SDW or a CDW.

Fig. 6 shows the first transport measurements performed on a single crystal by the IBM group. The first striking feature is the large anisotropy: $\rho_c$ is larger than $\rho_{ab}$ by a factor of 100. This anisotropy is seen to be even
Figure 6: First measurements of anisotropic resistivity made by the IBM group. The material was YBCO. Taken from [4]

larger in other systems, sometimes as high as $10^4$. Another interesting feature is the temperature dependence of the measurements: $\rho_{ab}$ is very linear from the transition temperature up to 300 K. In further measurements, this linearity was seen to extend above 1000 K. The first thing to note is that we see no sign of the $T^2$ dependence we would expect to see in a system with such large electron-electron interactions.

One may be tempted to attribute the linear behavior to phonon scattering. Since the Debye temperature is around 400 K in these materials [12], we will not expect to see any non-linearity from phonons until as low a temperatures as 40 or 50 K, which is well below the critical temperature in Fig. 6.

Fig. 7 shows resistance data from a sample of BISCO with a depressed $T_c$ of 7 K. Note that the sample shows very linear behavior extending all
the way down to 7 K, well below the temperature at which we should start seeing a $T^5$ contribution from phonons. The conclusion is that the linear $T$ behavior is not coming from phonons.

Another striking feature is that $\rho_{ab}$ does not seem to be affected by charged impurity scattering. Superconductivity in these materials only occurs when they are moderately doped, $x \approx 0.15$. The ionized donors generally lie in the interstitial layers about 4 Å from the CuO planes. From this, overestimating screening, Anderson argues ([4] pg. 63) that one should obtain a residual resistivity typically a factor of 10 or 100 of what is usually seen in the data. This point of view is also supported by the one exception to the disordered nature of the dopants, seen in the material YBa$_2$Cu$_3$O$_7$, the “stoichiometric” superconductor, shown in Fig. 6. Fig. 8 shows resistance data for optimally doped LSCO, which has a high degree of dopant disorder. Comparing Figs. 6, 7, and 8, we see that despite the large range
of impurity concentrations, the resistivities of these are all comparable, on the order of 100 \( \mu \Omega \). This suggests that charge carriers in the “normal” state are insensitive to scattering from charged impurities.

3 “The Dogmas”

In this section, I will provide an overview of what Anderson has called the “Central Dogmas” of high temperature superconductivity. The focus will be justifying the Dogma’s from experimental results, emphasizing the fact that they are not linked to any one specific theory, but are instead intended to constrain the Hilbert space of possible theories.

For this reason, I have decided to delay the mention of “Dogma VI”,

Figure 8: Resistance measurements in optimally doped \((\text{La-Sr})_2\text{CuO}_4\). From [4].
as it is too closely tied to Anderson’s idea’s of the mechanism to be justly described as a Dogma. I have also included a supplementary Dogma, which has to do with general considerations about the superconducting state.

**Dogma I:** All the relevant carriers of *both* spin and electricity derive from the hybridized O 2p - Cu 3d_{x^2−y^2} antibonding orbital in the CuO planes.

This is justified by our arguments from Section 2.2. The consequences of this dogma are that we should ignore theories based the chains of the YBCO compound, and that we can restrict ourselves to one-band theories, with the electronic structure of the interstitial layers being unimportant. This is well supported by the fact that the in-plane resistivity in the normal state only varies by a factor of 2 or so through all of the high $T_c$ compounds, which have vastly different interlayer structure. Note that this Dogma is not meant to imply we should completely ignore the interlayer structure: it simply implies that the charge carriers and relevant bands are in the planes.

**Dogma II:** Magnetism and high-$T_c$ superconductivity are closely related in a very specific sense: the electrons which exhibit magnetism are the same electrons that carry charge.

Any theory of high-$T_c$ cannot simply ignore the magnetic properties of the cuprates. This can be used to discount theories which are based on the Hubbard model with inappropriate parameters, which do not lead to the AF state at zero doping. In particular, it also discounts theories that don’t question where the magnetism comes from, but simply put it into the theory ad-hoc: the exotic magnetism in the cuprates must come from the charge
carriers themselves.

**Dogma III:** The dominant interaction are repulsive and there energy scales are all large. In terms of the Hubbard model, we must consider $U$ large but not infinite.

The statements of the above three dogmas are very general, and are well accepted today in the field of high $T_c$. They all point to one important route we should pursue on the theoretical front: to begin to understand high $T_c$, we must first understand the properties of the 2D, single-band, repulsive (but not too big) $U$ Hubbard model.

The next two of Anderson’s Dogmas are much more controversial. Essentially, they postulate properties of the solutions of this 2D Hubbard model from the observed behavior of the cuprates.

**Dogma IV:** The “normal” metal above $T_c$ is the solution of the one band Hubbard model from Dogma III, and is not a Fermi Liquid, in the sense that $Z=0^1$. It’s properties are similar to those the 1D Luttinger liquid.

The first experimental evidence is the enormous anisotropic conductivity. As we discussed in Section 2.5, this is not at all expected from Fermi liquid theory, where the conductivity is not affected by renormalizations of the effective mass. Genuinely tiny c-axis hopping matrix elements can lead to incoherent motion in the third dimension, but assuming such small matrix

\footnote{Z, the quasiparticle reside, is the overlap of the quasiparticle wavefunction with the momentum states of the free electron gas. It also gives the value of the jump in the momentum distribution function at the Fermi surface.}

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elements is not consistent with their estimated values and the fact that they are large enough to provide coherent c-axis motion in the superconductor.

So how does having “highly-correlated, non-Fermi liquid” in the plane help matters? In this supposed non-Fermi liquid state, the low lying excitation are not electron-like quasiparticles, but instead separate spin and charge collective excitations. In the interstitial layers, the eigenstates are electrons. Thus in tunneling from the plane into the interstitial layers, we project our collective excitation onto a superposition of the electron-like quasiparticles of the interstitial layer. Lets suppose that our state looks something like:

$$|\psi\rangle = a_1^\dagger |0\rangle = \sum C_n c_n^\dagger |0\rangle$$  \hfill (6)

where $a_1^\dagger$ is the creation operator for our collective excitation in the first plane and the $c_n^\dagger$ are the creation operators for the quasiparticles of the interstitial layer. We then can propagate the quasiparticles through the interstitial layer using a set of transmission matrix elements for the $c_n^\dagger$: our state rotates in Hilbert space, and we have a new set of coefficients $C_n'$. Once we reach the next CuO plane, we find ourselves in a new state $|\psi'\rangle$. Projecting this back onto the collective eigenstates of the CuO plane, we have a state:

$$|\psi'\rangle = \sum C_n' c_n^\dagger |0\rangle = \sum A_m a_2^\dagger |0\rangle$$  \hfill (7)

Because our excitations are made up of a large number of single particle states, $C_n$ will have a broad distribution. When we reach the second layer, we will project $C_n'$ back onto the collective excitation basis. However, since our collective excitation bears no resemblance to a quasiparticle (ie. $Z = 0$),
when we do this, we will not find ourselves projected onto a single collective excitation state, and the coefficients $A_m$ will be broadly distributed. The state we end up in is not coherent, and decays quickly. By this type of argument, the presence of a non-Fermi liquid state in the planes can lead to incoherent c-axis motion, while not requiring infinitesimal coupling between the planes.

Another piece of supporting evidence is the behavior of $\rho_{ab}$, specifically the fact that it is small and does not seem to be affected by scattering from phonons or charged impurities. This is somewhat less mysterious when we think of charge and spin being carried by collective modes involving a many body wavefunction: as in a BCS superconductor, the current carrying states are rigid against single particle scattering events.

Furthermore, the details of transport in a 1D Luttinger liquid, which are quite complicated (see [4] chap. 5, and [2]), can also lead to a weak coupling to charged impurities. In the Luttinger liquid, the spinon modes have a fermionic character, and do actually have a type of a Fermi surface, represented by power law divergence of the momentum distribution instead of a discontinuity. Although we have said that spin and charge are separated, the flow of the two is not completely decoupled: in the end, the current entering and leaving the leads involves electrons, and so there must be some correlation in the flow of spin and charge. This is manifested in a “backflow” coupling of the two, a long range but strong effect. The picture is that current is carried by a spinon accompanied by a “backflow cloud” of holons. The idea is similar to the polaron quasiparticle, where an electron in a metal carries with it a “cloud” of phonons. In this way, the charge carriers
are spin excitations, and are much less strongly coupled to charge impurity scattering.

Finally, the fact that $Z = 0$ is well supported by inelastic scattering times from infrared data. The data suggests that there is only inelastic scattering and that it is linear in $\omega$ (see [4] chap. 3): this leads to

$$Im\Sigma \sim \hbar\omega$$

Using the Kramers-Kronig relations, we expect

$$Re\Sigma \sim w \ln w$$

and

$$Z = \frac{1}{1 - \frac{\partial\Sigma}{\partial\omega}} \to 0 \text{ as } \omega \to 0$$

This is also supported in some of the photoemission data.

**Dogma V:** The above state, the solution of the Hubbard model, is strictly two dimensional, coherent transport in the third dimension being blocked. This state is *not* superconducting and has no major interactions tending to make it so at temperatures near the observed $T_c$.

The strongest experimental arguments for this is the variability of $T_c$ with the interlayer structure and the *smallness* of $T_c$. Firstly, the fact that the coupling between the planes has such a large effect on $T_c$ tells us that 3D coupling must play a very important role in the onset of superconductivity. Also, the material essentially undergoes a 2D to 3D transition at $T_c$, em-
phasizing the 3D nature of the superconducting state: we would not naively expect this if we are treating the material as entirely two dimensional.

The other argument is that $T_c$ is too small to be associated with the parameters of the 2D Hubbard model. The appropriate values of the Hubbard model parameters are in the thousands of Kelvin temperature scale. It is indeed true that the properties of the planes is very material independent on these temperature scales, and it is only near the temperature scales of $T_c$ that the different interplane couplings affect the physics.

**Supplemental Dogma:** The superconducting condensate is a many body wavefunction that has a gap in the low lying excitations that is independent of the magnitude of $\mathbf{k}$, and involves time reversed pairing of states.

The first part of the statement is really a requirement of any state that carries supercurrent. The supercurrent is possible in superconductors because the gap moves with the Fermi surface: we can then induce a current in the system by simply shifted the entire Fermi-sea. It is stable against scattering because the gap has moved with it, and there are never any low-lying excitations to scatter into.

The postulate about the pairing is supported by the experiments done by a group at IBM [14] where a persistent current was demonstrated in a ring consisting of partly high-$T_c$ superconductor and partly conventional superconductor. This requires that the phenomenology of any theory of high-$T_c$ must be that of a BCS-type order parameter describing a singlet ($\uparrow\downarrow$) pair [15].
4  Anderson’s Roadmap to Superconductivity

4.1  The Last Dogma

As I mentioned at the beginning of section 3, the last of Anderson’s Dogmas contains the basis of his theory for the mechanism and microscopic details responsible for this high critical temperatures in the cuprates. It is as follows:

**Dogma VI:** Interlayer hopping together with the “confinement” of Dogma V is either the mechanism or at least a major contributor to the superconducting condensation energy.

The basic idea is that there is a large missing kinetic energy, on the order of $t^2/\Delta$, that is missing in the normal state owing to the fact that coherent motion of the electrons in the c-axis direction is blocked. This energy is regained in the superconducting state, which is three dimensional, and this is what drives the critical temperature to such a large value. As Anderson emphasizes, that this is a contribution to the condensate energy is not just a theoretical supposition but an experimental fact: we know that there must be a missing kinetic energy due to confinement to the planes, and we know that this is restored in the superconducting state.

Note that we have not restricted ourselves to any specific pairing mechanism: any pairing mechanism that will allow coherent tunneling between the layers will be able to gain access to this missing energy. As we shall see, Anderson proposes two pairing mechanisms that use interlayer hopping to pair excitations of the normal state, one based on a spin “superexchange” between adjacent planes, and one based on a coherent “pair hop-
ping” involving the simultaneous hopping of two normal state excitations to an adjacent plane. We are not restricted to these mechanisms, though, and the condensate may also gain energy through other pairing mechanisms through phonons or other modes. We can also now energetically favor pairing through a repulsive interaction, as we are not entirely relying on the pairing coupling to provide the energy of the condensate.

Finally, we note that Dogma VI is well supported by the increase of $T_c$ in structures that have stronger interlayer coupling.

4.2 Anderson’s Pairing Mechanisms

As the underlying paring mechanism for the cuprates, Anderson has proposed two terms that can lead to coherent tunneling of electron pairs between planes:

$$\mathcal{H}_{SE} = \sum_k \lambda_{SE}^{(k)} c_{k\uparrow}^\dagger (1) c_{-k\downarrow}^\dagger (2) c_{-k\downarrow} (1) c_{k\uparrow} (2)$$  (8)

$$\mathcal{H}_{PT} = \sum_k \lambda_{PT}^{(k)} c_{k\uparrow}^\dagger (1) c_{-k\downarrow}^\dagger (1) c_{-k\downarrow} (2) c_{k\uparrow} (2)$$  (9)

The Feynman diagrams for the processes are shown in Fig. 9. We can see that the superexchange term involves a swap of the momenta and spins of two electrons in adjacent planes. The pair tunneling term involves the scattering of two electrons in the first plane into two electron states in the second plane.

We note that in the first diagram, only spin is exchanged between the planes: no charge is transferred. In the second diagram, there is no net transfer of spin between the planes, and only charge is transferred. It is
Figure 9: Diagrams for the pairing terms: (a) the superexchange term, and (b) the pair tunneling term.

this that allows these tunneling processes to be coherent: the first one can be expressed as an exchange of spinons, while the second as an exchange of holons—ie. the tunneling of two electrons through these processes involves tunneling of states that are eigenstates of the CuO plane Hamiltonian. The reason why single particle tunneling was incoherent was that we were trying to tunnel both charge and spin at the same time, and there are no eigenstates in the plane that carry both.

5 The Future of High $T_c$

Having now examined an overview of the high $T_c$ problem, and gone through the arguments of one of the competing theories, we must sit down and ask ourselves: where do we go from here? What can I as a physicist do to advance our understanding of high temperature superconductivity?

Our discussion has been strongly biased towards Anderson’s line of thought. While I personally found that Anderson’s theory is fairly consistent and well
thought out, it is far from being complete. Some of the arguments need to be examined more carefully, particularly those leading to the last three Dogmas. It is nice to keep it in the back of one’s mind when trying to understand aspects of high temperature superconductors, but it would be foolish to completely commit oneself to any one theory at this stage.

Anderson’s ideas aside, though, from the theoretical point of view, there is one thing that is clear and evident: we must solve the 2D Hubbard model with realistic parameters. Whether this is done analytically or numerically, an understanding of the physics of this model is crucial if we want to even approach the superconductivity problem. Are the divergences of the vertices in 2D in fact significant enough to invalidate Fermi liquid theory? Does the solution for $U$ large but not infinite in fact have $Z = 0$? Does the system show spin-charge separation? What are the mechanisms of transport in the normal state? Are there any possible purely 2D mechanisms that arise in the 2D Hubbard model physics that could lead to the observed critical temperatures?

From an experimentalist’s perspective, we must simply do more experiments and make sure we do them properly. The consensus is now that all measurements should be done with high purity single crystals. There are many early experiments that would benefit from a reexamination using such samples. It is also crucial that experimental results should be presented in such a way that they stand on their own. Too many experimental papers have been rendered useless because the authors went to great efforts to squeeze the data into the context of some theory, leaving interpretation outside of this context nearly impossible. Finally, the cuprates have a very
rich phase diagram that is abundant with interesting and unique many body states of the electron liquid. There is much phase space to explore: charting it all carefully can only serve to deepen our understanding of the spectacular physics of the cuprates.

References


[12] For an extensive database of the properties of the cuprates, see http://www.ceramics.nist.gov/srd/hts/htsquery.htm

